

EXAMPLES OF NON-ALGEBRAIC CLASSES IN THE BROWN-PETERSON TOWER

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ABSTRACT. For every $n \geq 0$, we construct classes in the Brown-Peterson cohomology $BP\langle n \rangle$ of smooth projective complex algebraic varieties which are not in the image of the cycle map from the corresponding motivic Brown-Peterson cohomology. This generalizes the examples of Atiyah and Hirzebruch to all finite levels in the Brown-Peterson tower.

1. INTRODUCTION

Let X be a smooth projective complex algebraic variety. Let

$$\text{cl}: CH^*(X) \rightarrow H^{2*}(X; \mathbb{Z})$$

be the cycle map from Chow groups to the singular cohomology of the space $X(\mathbb{C})$ of complex points of X . Recall that classes in the image of cl are called *algebraic* and that all algebraic classes are contained in the subgroup of integral Hodge classes. However, it is well known that, in general, not all integral Hodge classes are algebraic. There are basically two types of examples of non-algebraic Hodge classes, one which can be detected by topological methods and one which cannot be detected by topological invariants. In [2], Atiyah and Hirzebruch used the Atiyah-Hirzebruch spectral sequence to construct an obstruction for elements being in the image of cl and provided, for every prime p , examples of non-algebraic p -torsion classes using the Godeaux-Serre construction of varieties associated to finite groups. In [19], Totaro showed that the Atiyah-Hirzebruch obstruction can also be explained by the fact that cl factors through the natural map $\bar{\vartheta}: MU^{2*}(X) \otimes_{MU^*} \mathbb{Z} \rightarrow H^{2*}(X; \mathbb{Z})$, where $MU^*(X)$ denotes the complex cobordism of the space $X(\mathbb{C})$. A crucial fact is that the map $\bar{\vartheta}$ is neither surjective nor injective in general. In [8], Kollár provided examples of varieties X and non-torsion classes $\alpha \in H^4(X; \mathbb{Z})$ which are not algebraic while a multiple of α is algebraic. In [16], Soulé and Voisin explain these examples in detail and show that there is no locally constant invariant that can detect these examples. In particular, for those varieties, $\bar{\vartheta}$ is surjective. Moreover, Soulé and Voisin construct other types of non-algebraic torsion classes which cannot be explained by the obstruction of Atiyah-Hirzebruch and Totaro. We briefly discuss some more examples in Remark 4.7.

One way to define the cycle map for smooth complex varieties is to interpret it as the natural map

$$\text{cl}: H_{\mathcal{M}}^{2*,*}(X; \mathbb{Z}) \rightarrow H^{2*}(X; \mathbb{Z})$$

from motivic to singular cohomology induced by topological realization. The advantage of this definition for our purposes is that it immediately generalizes to other motivic cohomology theories. It is then a natural question what we can say about the image of the corresponding cycle map for other motivic theories.

The purpose of this paper is to show that the examples of Atiyah and Hirzebruch can be generalized to produce non-algebraic classes in all levels of the tower of Brown-Peterson cohomology theories which interpolate between singular cohomology and the p -localization of complex cobordism.

To be more precise, for a prime p , let $BP^*(-)$ denote the Brown-Peterson cohomology for p (see [3] and [14]). For an integer $n \geq 0$, let $BP\langle n \rangle$ be the associated intermediate theory studied in [20]. These cohomology theories fit into a sequence

$$BP^*(X) \rightarrow \cdots \rightarrow BP\langle n \rangle^*(X) \rightarrow \cdots \rightarrow BP\langle 0 \rangle^*(X) = H^*(X; \mathbb{Z}_{(p)}).$$

For every p and n , there is a corresponding motivic Brown-Peterson cohomology which we denote by $BP\langle n \rangle_{\mathcal{M}}^{*,*}(X)$ (see [5, §6.4] and [10, §3]). For every i and j , the topological realization functor from the motivic to the classical stable homotopy category induces a natural homomorphism

$$BP\langle n \rangle_{\mathcal{M}}^{i,j}(X) \rightarrow BP\langle n \rangle^i(X)$$

to the Brown-Peterson cohomology of the space $X(\mathbb{C})$.

For given p and n , we write $w(n) := p^n + p^{n-1} + \cdots + p + 1$. Our main result is the following.

Theorem. *For every prime p and every integer $n \geq 0$, there is a smooth projective complex algebraic variety X and an element $b_n \in BP\langle n \rangle^{2w(n)+2}(X)$ which is not in the image of the map*

$$\text{cl}_n: BP\langle n \rangle_{\mathcal{M}}^{2w(n)+2, w(n)+1}(X) \rightarrow BP\langle n \rangle^{2w(n)+2}(X).$$

The idea of the proof of the theorem and the structure of the paper can be summarized as follows. We start with the fundamental stable cofiber sequence

$$\Sigma^{2(p^n-1)} BP\langle n \rangle \rightarrow BP\langle n \rangle \rightarrow BP\langle n-1 \rangle.$$

It yields a well known obstruction for elements being in the image of the map $\rho_n^{n+1}: BP\langle n+1 \rangle^*(X) \rightarrow BP\langle n \rangle^*(X)$ provided by Milnor operations in mod p -cohomology $H^*(X; \mathbb{F}_p)$. However, it also provides a tool to lift elements from $H^*(X; \mathbb{F}_p)$ to $BP\langle n \rangle^{*, 2w(n)-n-1}(X)$. We apply this observation to the elementary abelian p -group $G_{n+3} := (\mathbb{Z}/p)^{n+3}$, for any prime number p , and construct explicit elements in $BP\langle n \rangle^*(BG_{n+3})$ which are not contained in the image of the map $\rho_n^{n+1}: BP\langle n+1 \rangle^*(BG_{n+3}) \rightarrow BP\langle n \rangle^*(BG_{n+3})$. Finally, we use these elements to prove the theorem for a Godeaux-Serre variety X associated to G_{n+3} .

Our argument relies on the fact that ρ_n is not surjective in high degrees for the particular variety we consider. Wilson showed in [20] that, for any finite complex X and all $k \leq 2w(n)$, the map $\rho_n: BP^k(X) \rightarrow BP\langle n \rangle^k(X)$ is surjective. The degree in which the examples of the theorem occur is hence minimal for our argument.

In Remark 4.6, we will briefly discuss a different type of argument for complex cobordism which the surjectivity of the map $MU^{2*}(X) \rightarrow H^{2*}(X; \mathbb{Z})$ for Kollár's examples.

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2. OBSTRUCTIONS AND LIFTINGS

Let p be a prime number and $n \geq 0$ be an integer. Let BP denote the spectrum representing Brown-Peterson cohomology for p defined in [3], and let $BP\langle n \rangle$ be the spectrum representing the associated intermediate theory for p and n studied by Wilson in [20]. For $n = 0$, one has $BP\langle 0 \rangle = H\mathbb{Z}_{(p)}$, the Eilenberg-MacLane spectrum for $\mathbb{Z}_{(p)}$, and, for $n = -1$, we use the notation $BP\langle -1 \rangle := H\mathbb{F}_p$ for the mod p -Eilenberg-MacLane spectrum. For every $j > n \geq -1$, these theories are connected by canonical maps

$$\rho_n: BP \rightarrow BP\langle n \rangle \text{ and } \rho_n^j: BP\langle j \rangle \rightarrow BP\langle n \rangle.$$

The coefficient rings are given by the polynomial algebras $BP^* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ and $BP\langle n \rangle^* \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$. The effect of the maps ρ_n and ρ_n^j is to send all v_i with $i > n$ to 0.

Recall that, for every $n \geq 1$, there is a stable cofiber sequence

$$\Sigma^{2(p^n-1)} BP\langle n \rangle \xrightarrow{v_n} BP\langle n \rangle \xrightarrow{\rho_{n-1}^n} BP\langle n-1 \rangle \xrightarrow{q_n} \Sigma^{2(p^n-1)+1} BP\langle n \rangle.$$

For every space X , this sequence induces a natural exact sequence

$$BP\langle n \rangle^{i+2(p^n-1)}(X) \xrightarrow{v_n} BP\langle n \rangle^i(X) \xrightarrow{\rho_{n-1}^n} BP\langle n-1 \rangle^i(X) \xrightarrow{q_n} BP\langle n \rangle^{i+2p^n-1}(X)$$

where, by abuse of notation, we denote the induced maps on cohomology groups by the same symbols.

By [20, Proposition 1.7] (see also [12, Lemma 2.4] and [18, Proposition 4-4]), the map $q_n: BP\langle n-1 \rangle^i(X) \rightarrow BP\langle n \rangle^{i+2p^n-1}(X)$ corresponds, possibly up to a sign, to the n th Milnor operation Q_n in mod p -cohomology in the sense that there is a commutative diagram

$$(1) \quad \begin{array}{ccccc} BP\langle n \rangle^*(X) & \xrightarrow{\rho_{n-1}^n} & BP\langle n-1 \rangle^i(X) & \xrightarrow{q_n} & BP\langle n \rangle^{i+2p^n-1}(X) \\ & \searrow \rho_{n-1}^n & \downarrow \rho_{n-1}^{n-1} & & \downarrow \rho_{n-1}^n \\ & & H^i(X; \mathbb{F}_p) & \xrightarrow{\pm Q_n} & H^{i+2p^n-1}(X; \mathbb{F}_p). \end{array}$$

Hence, since the top row of diagram (1) is exact, Q_n yields an obstruction to lifting an element from $BP\langle n-1 \rangle^*(X)$ to $BP\langle n \rangle^*(X)$ via ρ_{n-1}^n .

On the other hand, we can use the maps q_n to produce explicit classes in $BP\langle n \rangle^*(X)$. Recall that the degree of Q_i is $|Q_i| = 2p^i - 1$ and hence

$$\sum_{i=0}^n |Q_i| = \sum_{i=0}^n (2p^i - 1) = 2w(n) - n - 1$$

where we set $w(n) := p^n + \dots + 1$. Successive composition of the maps q_i yields a diagram which commutes (possibly up to sign)

$$\begin{array}{ccccccc} H\mathbb{F}_p & \xrightarrow{q_0} & \Sigma^{|Q_0|} BP\langle 0 \rangle & \xrightarrow{q_1} & \Sigma^{|Q_0|+|Q_1|} BP\langle 1 \rangle & \xrightarrow{q_2} \dots & \xrightarrow{q_n} \Sigma^{2w(n)-n-1} BP\langle n \rangle \\ \downarrow \pm Q_{n+1} \dots Q_0 & & & & & & \downarrow q_{n+1} \\ \Sigma^{2w(n+1)-n-2} H\mathbb{F}_p & \xleftarrow{\rho_{-1}^{n+1}} & & & & & \Sigma^{2w(n+1)-n-2} BP\langle n+1 \rangle. \end{array}$$

Hence, for every X , there is a commutative diagram
(2)

$$\begin{array}{ccccc}
 BP\langle n+1 \rangle^{*+2w(n)-n-1}(X) & \xrightarrow{\rho_n^{n+1}} & BP\langle n \rangle^{*+2w(n)-n-1}(X) & \xrightarrow{q_{n+1}} & BP\langle n+1 \rangle^{*+2w(n+1)-n-2}(X) \\
 & & \uparrow q_n \cdots q_0 & & \downarrow \rho_{n-1}^{n+1} \\
 & & H^*(X; \mathbb{F}_p) & \xrightarrow{\pm Q_{n+1} \cdots Q_0} & H^{*+2w(n+1)-n-2}(X; \mathbb{F}_p)
 \end{array}$$

where the top row is exact. This yields the following criterion.

Lemma 2.1. *If $x \in H^*(X; \mathbb{F}_p)$ satisfies $Q_{n+1} \cdots Q_0(x) \neq 0$, then*

$$q_n \cdots q_0(x) \in BP\langle n \rangle^{*+2w(n)-n-1}(X)$$

is a non-trivial element which is not contained in the image of

$$\rho_n^{n+1}: BP\langle n+1 \rangle^*(X) \rightarrow BP\langle n \rangle^*(X).$$

Remark 2.2. In [20], Wilson showed that, for every space X and integer $n \geq 0$, the natural homomorphism

$$BP^k(X) \rightarrow BP\langle n \rangle^k(X)$$

is surjective for $k \leq 2w(n)$. Hence the lowest even degree in which we can hope to find classes in $BP\langle n \rangle^k(X)$ which cannot be lifted to $BP\langle n+1 \rangle^k(X)$ is $k = 2w(n)+2$. This means that an element $x \in H^*(X; \mathbb{F}_p)$ with the properties in Lemma 2.1 must be of degree at least $n+3$.

3. $BP\langle n \rangle$ -CLASSES FOR ELEMENTARY ABELIAN p -GROUPS

In this section, we look at the $BP\langle n \rangle$ -cohomology of the classifying spaces of elementary abelian p -groups. The generalized cohomology of such spaces is a well studied subject in the literature (see e.g. [12] and [17] for the case $BP\langle n \rangle$). The goal of this section is merely to specify concrete elements with the properties needed to apply Lemma 2.1. The cases $p = 2$ or p odd are very similar. However, for the convenience of the reader, we provide the computations for both cases separately. To simplify the notation, for G a group and h a cohomology theory, we write $h^*(G)$ for $h^*(BG_+)$, where BG denotes the classifying space of G .

3.1. Non-liftable $BP\langle n \rangle$ -classes for $p = 2$. We start with the case $p = 2$. Recall that Milnor's operations

$$Q_n: H^i(X; \mathbb{F}_2) \rightarrow H^{i+2^{n+1}-1}(X; \mathbb{F}_2)$$

are defined inductively in terms of Steenrod squares by

$$\begin{aligned}
 Q_0 &= Sq^1, \\
 Q_{n+1} &= Sq^{2^{n+1}} Q_n + Q_n Sq^{2^{n+1}}.
 \end{aligned}$$

A very convenient fact about the Q_n 's is that they are primitive elements of the Steenrod algebra, i.e.,

$$(3) \quad Q_n(xy) = Q_n(x)y + xQ_n(y).$$

Let G_k be the k -fold product of $\mathbb{Z}/2$, i.e., $G_k = (\mathbb{Z}/2)^k$. The \mathbb{F}_2 -cohomology of G_k is given by the formula

$$H^*((\mathbb{Z}/2)^k; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots, x_k].$$

Lemma 3.1. *Let x be a polynomial generator of $H^*((\mathbb{Z}/2)^k; \mathbb{F}_2)$. For $i \geq 0$, the Milnor operation Q_i acts on x by*

$$(4) \quad Q_i(x) = x^{2^{i+1}}$$

$$(5) \quad Q_i(x^{2^n}) = 0 \text{ for all } n \geq 1.$$

Proof. For $i = 0$, we have $Q_0(x) = Sq^1(x) = x^2$ and

$$Q_0(x^{2^n}) = Sq^1 x^{2^n} = \binom{2n}{1} x^{2n+1} = 2n x^{2n+1} = 0$$

for $n \geq 1$, since we are working modulo 2. For $i \geq 1$, we proceed by induction using (3). For $n = 0$, we get

$$Q_i(x) = (Sq^{2^i} Q_{i-1} + Q_{i-1} Sq^{2^i})(x) = Sq^{2^i}(x^{2^i}) + Q_{i-1}(0) = x^{2^{i+1}}.$$

For $n \geq 1$, we have

$$Q_i(x^{2^n}) = (Sq^{2^i} Q_{i-1} + Q_{i-1} Sq^{2^i})(x^{2^n}) = Sq^{2^i} Q_{i-1}(x^{2^n}) + Q_{i-1} Sq^{2^i}(x^{2^n}) = 0$$

by the induction hypothesis on Q_{i-1} . \square

Lemma 3.2. *For $m \leq k$, let x_1, \dots, x_m be distinct polynomial generators of $H^*((\mathbb{Z}/2)^k; \mathbb{F}_2)$. The effect of the iterated Milnor operations on the product $x_1 \cdots x_m$ is given by*

$$(6) \quad Q_n Q_{n-1} \cdots Q_1 Q_0(x_1 \cdots x_m) = 0 \text{ for } m \leq n$$

$$(7) \quad Q_n Q_{n-1} \cdots Q_1 Q_0(x_1 \cdots x_m) = \sum x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m} \text{ for } m \geq n+1$$

where the sum is taken over all permutations (j_1, \dots, j_m) of the set of m numbers $\{2^{n+1}, 2^n, \dots, 2^1, 1, \dots, 1\}$ with $m - (n+1)$ many 1's. In particular, we get

$$Q_n Q_{n-1} \cdots Q_1 Q_0(x_1 \cdots x_m) \neq 0 \text{ for } m \geq n+1.$$

Proof. For $n = 0$, Lemma 3.1 and equation (3) imply

$$Q_0(x_1 \cdots x_m) = x_1^2 x_2 \cdots x_m + x_1 x_2^2 \cdots x_m + \cdots + x_1 x_2 \cdots x_m^2$$

where in each summand there is exactly one exponent equal to 2 and all others are equal to 1. This proves the case $n = 0$. For $n = 1$ and $m = 1$, we have

$$Q_1 Q_0(x_1) = Q_1(x_1^2) = 0$$

by (5). Hence equations (6) and (7) hold for $n = 0$ and the case $n = 1$ and $m = 1$. For $n \geq 1$ and $m \geq 1$, we proceed by induction. If $m \leq n-1$, then $Q_{n-1} \cdots Q_1 Q_0(x_1 \cdots x_m) = 0$ by the induction hypothesis. So assume $m \geq n$. By the induction hypothesis, we have

$$Q_n Q_{n-1} \cdots Q_1 Q_0(x_1 \cdots x_m) = Q_n \left(\sum x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m} \right)$$

where the sum is taken over all permutations (j_1, \dots, j_m) of the set of m numbers $\{2^n, 2^{n-1}, \dots, 2^1, 1, \dots, 1\}$. For each summand, equation (3) implies

$$\begin{aligned} Q_n(x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}) &= Q_n(x_1^{j_1})(x_2^{j_2} \cdots x_m^{j_m}) + x_1^{j_1} Q_n(x_2^{j_2} \cdots x_m^{j_m}) \\ &= Q_n(x_1^{j_1})(x_2^{j_2} \cdots x_m^{j_m}) + x_1^{j_1} (Q_n(x_2^{j_2})(x_3^{j_3} \cdots x_m^{j_m}) + x_2^{j_2} Q_n(x_3^{j_3} \cdots x_m^{j_m})) \\ &\vdots \end{aligned}$$

$$(8) \quad = Q_n(x_1^{j_1}) x_2^{j_2} \cdots x_m^{j_m} + x_1^{j_1} Q_n(x_2^{j_2}) x_3^{j_3} \cdots x_m^{j_m} + \cdots + x_1^{j_1} \cdots x_{m-1}^{j_{m-1}} Q_n(x_m^{j_m}).$$

If $m = n$, then each (j_1, \dots, j_n) is a bijection of $\{1, \dots, m\}$ with $\{2^n, 2^{n-1}, \dots, 2^2, 2^1\}$. In particular, we have $j_i \geq 2$ for all $i = 1, \dots, n$. By formula (5), this implies that $Q_n(x_i^{j_i}) = 0$ for $i = 1, \dots, n$. Hence, if $m = n$, line (8) is equal to zero. This finishes the proof of equation (6).

If $m \geq n + 1$, then we have $Q_n(x_i^{j_i}) = 0$ if $j_i \geq 2$ by (5) and $Q_n(x_i^{j_i}) = x_i^{2^{n+1}}$ if $j_i = 1$ by (4). This implies that, for a fixed permutation (j_1, \dots, j_m) of the set of m numbers $\{2^n, 2^{n-1}, \dots, 2^1, 1, \dots, 1\}$, the element $Q_n(x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m})$ is the sum of terms of the form $x_1^{j_1} \cdots x_{i-1}^{j_{i-1}} x_i^{2^{n+1}} x_{i+1}^{j_{i+1}} \cdots x_m^{j_m}$, one summand for each i with $j_i = 1$. Taking the sum over all permutations (j_1, \dots, j_m) of the set of m numbers $\{2^n, 2^{n-1}, \dots, 2^1, 1, \dots, 1\}$ ($m - n$ many 1's), then yields

$$Q_n Q_{n-1} \cdots Q_1 Q_0(x_1 \cdots x_m) = \sum x_1^{j'_1} x_2^{j'_2} \cdots x_m^{j'_m}$$

where the sum is taken over all bijections (j'_1, \dots, j'_m) of $\{1, \dots, m\}$ with the set of m numbers $\{2^{n+1}, 2^n, \dots, 2^1, 1, \dots, 1\}$ ($m - n - 1$ many 1's). This finishes the proof of equation (7) and the lemma. \square

Recall that the degree of Q_i is $|Q_i| = 2^{i+1} - 1$ and hence

$$\sum_{i=0}^n |Q_i| = \sum_{i=0}^n 2^{i+1} - 1 = 2^{n+2} - 2 - (n+1) = 2^{n+2} - 3 - n.$$

Proposition 3.3. *Let n be an integer ≥ 0 and let k and m be integers such that $k \geq m \geq n + 2$. Let $G_k = (\mathbb{Z}/2)^k$ and x_1, \dots, x_m be distinct polynomial generators of $H^*((\mathbb{Z}/2)^k; \mathbb{F}_2)$. Then the element $q_n \cdots q_0(x_1 \cdots x_m)$ is nontrivial in the group $BP\langle n \rangle^{m+2^{n+2}-3-n}(G_k)$ and is not contained in the image of the map*

$$\rho_n^{n+1}: BP\langle n+1 \rangle^{m+2^{n+2}-3-n}(G_k) \rightarrow BP\langle n \rangle^{m+2^{n+2}-3-n}(G_k).$$

Proof. We know by diagram (2) that

$$\rho_{-1}^{n+1} q_{n+1}(q_n \cdots q_0(x_1 \cdots x_m)) = Q_{n+1} \cdots Q_0(x_1 \cdots x_m) \text{ in } H^{m+2^{n+3}-4-n}(G_k; \mathbb{F}_2).$$

By Lemma 3.2, we know that $Q_{n+1} \cdots Q_0(x_1 \cdots x_m)$ nontrivial if $m \geq n + 2$. The assertion then follows from Lemma 2.1. \square

Remark 3.4. For our application to algebraic varieties, we are interested in elements in even degree in $BP\langle n \rangle^*(G_k)$. Hence the minimal values for k and m such that $q_n \cdots q_0(x_1 \cdots x_m)$ is nontrivial is $k = m = n + 3$. In this case we have

$$q_n \cdots q_0(x_1 \cdots x_{n+3}) \neq 0 \text{ in } BP\langle n \rangle^{2^{n+2}}(G_{n+3})$$

and cannot be lifted to $BP\langle n+1 \rangle^{2^{n+2}}(G_{n+3})$.

3.2. Non-liftable $BP\langle n \rangle$ -classes for odd primes. In this section, let p be an odd prime. Let G_k be the k -fold product of \mathbb{Z}/p , i.e., $G_k = (\mathbb{Z}/p)^k$. The \mathbb{F}_p -cohomology of G_k is given by

$$(9) \quad H^*((\mathbb{Z}/p)^k; \mathbb{F}_p) \cong \Lambda(x_1, \dots, x_k) \otimes \mathbb{F}_p[y_1, \dots, y_k]$$

with $|x_i| = 1$ and $|y_i| = 2$ (and $x_i^2 = 0$) for $i = 1, \dots, k$. The Bockstein homomorphism

$$\beta: H^*(G_k; \mathbb{Z}/p) \rightarrow H^{*+1}(G_k; \mathbb{Z}/p).$$

sends x_i to y_i , i.e., $\beta(x_i) = y_i$ for $i = 1, \dots, k$. The Milnor operations Q_n in the mod p -Steenrod algebra can be defined recursively by

$$\begin{aligned} Q_0 &= \beta, \\ Q_{n+1} &= P^{p^n} Q_n - Q_n P^{p^n} \end{aligned}$$

where P^i is the i th reduced p th power operation. The Q_n are primitive elements, i.e., $Q_n(xy) = Q_n(x)y + (-1)^{|x| \cdot |Q_n|} x Q_n(y)$. Since the degree of Q_n is always odd, this means

$$(10) \quad Q_n(xy) = Q_n(x)y + (-1)^{|x|} x Q_n(y).$$

Lemma 3.5. *The action of the Milnor operations on the generators of $H^*((\mathbb{Z}/p)^k; \mathbb{F}_p)$ is given by*

$$(11) \quad Q_n(x_i) = y_i^{p^n}$$

$$(12) \quad Q_n(y_i) = 0.$$

Proof. For $n = 0$, we have $Q_0(x_i) = \beta(x_i) = y_i$, and $Q_0(y_i) = \beta(\beta(x_i)) = 0$. For $n \geq 1$, we proceed by induction. For x_i , we get

$$\begin{aligned} Q_n(x_i) &= (P^{p^{n-1}} Q_{n-1} - Q_{n-1} P^{p^{n-1}})(x_i) \\ &= P^{p^{n-1}} Q_{n-1}(x_i) - Q_{n-1} P^{p^{n-1}}(x_i) \\ &= P^{p^{n-1}}(y_i^{p^{n-1}}) - Q_{n-1}(0) = (y_i^{p^{n-1}})^p \\ &= y_i^{p^n}. \end{aligned}$$

For y_i , we get

$$\begin{aligned} Q_n(y_i) &= (P^{p^{n-1}} Q_{n-1} - Q_{n-1} P^{p^{n-1}})(y_i) \\ &= P^{p^{n-1}} Q_{n-1}(y_i) - Q_{n-1} P^{p^{n-1}}(y_i) \\ &= 0 \end{aligned}$$

since Q_{n-1} acts trivially on y_i by the induction hypothesis (and $P^{p^{n-1}}(y_i) = 0$ for $n \geq 2$). \square

In order to facilitate the bookkeeping in the next lemma, we set

$$(13) \quad y_i^{\frac{1}{2}} := x_i \text{ for } i = 1, \dots, k$$

in $H^*((\mathbb{Z}/p)^k; \mathbb{F}_p) = \Lambda(x_1, \dots, x_k) \otimes \mathbb{F}_p[y_1, \dots, y_k]$.

Lemma 3.6. *With notation as in (9) and (13) and $m \leq k$, we have*

$$\begin{aligned} Q_n Q_{n-1} \cdots Q_0(x_1 \cdots x_m) &= 0 \text{ for } m \leq n \\ Q_n Q_{n-1} \cdots Q_0(x_1 \cdots x_m) &= \sum (-1)^{\rho(j)} y_1^{j_1} y_2^{j_2} \cdots y_m^{j_m} \text{ for } m \geq n+1 \end{aligned}$$

where the sum is taken over all bijections $j = (j_1, \dots, j_m)$ of the set $\{1, \dots, m\}$ with the set of m numbers $\{p^n, p^{n-1}, \dots, p, 1, \frac{1}{2}, \dots, \frac{1}{2}\}$ with $m - (n+1)$ many $\frac{1}{2}$'s, and $\rho(j)$ is the sum over the numbers a_t , for $t \in \{1, \dots, n\}$, defined as follows: let $i(t)$ be the index with $j_{i(t)} = p^t$; then a_t is the number of indices $s \in \{1, \dots, m\}$ with $s < i(t)$ and either $j_s = \frac{1}{2}$ or $j_s > p^t$.

In particular, we get

$$Q_n Q_{n-1} \cdots Q_1 Q_0(x_1 \cdots x_m) \neq 0 \text{ for } m \geq n+1.$$

Proof. For $n = 0$, we have

$$Q_0(x_1 \cdots x_m) = y_1 x_2 \cdots x_m - x_1 y_2 x_3 \cdots x_m + \cdots + (-1)^m x_1 x_2 \cdots y_m.$$

Keeping notation (13) in mind, this proves the assertion for $n = 0$.

For $n = 1$ and $m = 1$, we have $Q_1 Q_0(x_1) = Q_1(y_1) = 0$ by (12). For $n \geq 1$ and $m \geq 1$, we proceed by induction. If $m \leq n - 1$, then $Q_{n-1} \cdots Q_1 Q_0(x_1 \cdots x_m) = 0$ by the induction hypothesis. So assume $m \geq n$. By the induction hypothesis, we have

$$Q_n Q_{n-1} \cdots Q_0(x_1 \cdots x_m) = Q_n \left(\sum (-1)^{\rho(j)} y_1^{j_1} y_2^{j_2} \cdots y_m^{j_m} \right)$$

where the sum is taken over all bijections $j = (j_1, \dots, j_m)$ of the set $\{1, \dots, m\}$ with the set of m numbers $\{p^{n-1}, \dots, p, 1, \frac{1}{2}, \dots, \frac{1}{2}\}$ (with $m - n$ many $\frac{1}{2}$'s).

If $m = n$, then each (j_1, \dots, j_n) is a permutation of the set $\{p^{n-1}, \dots, p, 1\}$, or in other words, there is no x_i left. Then (12) implies that all summands vanish under Q_n . This proves the assertion for $m = n$.

Now we assume $m \geq n + 1$. Let j be a fixed bijection from $\{1, \dots, m\}$ to $\{p^{n-1}, \dots, p, 1, \frac{1}{2}, \dots, \frac{1}{2}\}$. By formula (10), applying Q_n to $y_1^{j_1} y_2^{j_2} \cdots y_m^{j_m}$ yields new summands, one for each $i \in \{1, \dots, m\}$ with $j_i = \frac{1}{2}$, of the form

$$(-1)^a y_1^{j_1} \cdots y_{i-1}^{j_{i-1}} y_i^{p^n} y_{i+1}^{j_{i+1}} \cdots y_m^{j_m}$$

where a is given by the number of indices $s \in \{1, \dots, m\}$ with $s < i$ and $j_s = \frac{1}{2}$. Now taking the sum over all bijections j from $\{1, \dots, m\}$ to $\{p^{n-1}, \dots, p, 1, \frac{1}{2}, \dots, \frac{1}{2}\}$, we obtain that

$$Q_n \cdots Q_0(x_1 \cdots x_m) = \sum (-1)^{\rho(j')} y_1^{j'_1} y_2^{j'_2} \cdots y_m^{j'_m}$$

where the sum is now over all bijections j' of $\{1, \dots, m\}$ with $\{p^n, \dots, p, 1, \frac{1}{2}, \dots, \frac{1}{2}\}$. \square

Recall $|Q_i| = 2p^i - 1$ and $\sum_{i=0}^n |Q_i| = \sum_{i=0}^n 2p^i - 1 = 2w(n) - n - 1$ where we write $w(n) := p^n + \cdots + 1$.

Proposition 3.7. *Let n be an integer ≥ 0 , and let k and m be integers such that $k \geq m \geq n + 2$. Let $G_k = (\mathbb{Z}/p)^k$ and x_1, \dots, x_m be distinct exterior algebra generators of $H^*((\mathbb{Z}/p)^k; \mathbb{F}_p)$ as in formula (9). Then the element $q_n \cdots q_0(x_1 \cdots x_m)$ is nontrivial in the group $BP\langle n \rangle^{m+2w(n)-n-1}(G_k)$ and is not contained in the image of the map*

$$\rho_n^{n+1}: BP\langle n+1 \rangle^{m+2w(n)-n-1}(G_k) \rightarrow BP\langle n \rangle^{m+2w(n)-n-1}(G_k).$$

Proof. By diagram (2), we know

$$\rho_{-1}^{n+1} q_{n+1}(q_n \cdots q_0(x_1 \cdots x_m)) = \pm Q_{n+1} \cdots Q_0(x_1 \cdots x_m) \in H^{m+2w(n+1)-n-2}(G_k; \mathbb{F}_p).$$

By Lemma 3.6, $Q_{n+1} \cdots Q_0(x_1 \cdots x_m)$ is nontrivial if $m \geq n + 2$. The assertion then follows from Lemma 2.1. \square

By [12, Corollary 7.10], we know that any element in $BP\langle n \rangle^*(G_k)$ which is not in the image of the map $BP^*(G_k) \rightarrow BP\langle n \rangle^*(G_k)$ is in the image of the map

$$q_n \cdots q_0: H^*(G_k; \mathbb{F}_p) \rightarrow BP\langle n \rangle^{*+2w(n)-n-1}(G_k).$$

The point of Propositions 3.3 and 3.7 is that we specify concrete nontrivial elements in this image that we can use for our application in the next section.

4. NON-ALGEBRAIC CLASSES IN $BP\langle n \rangle$ -COHOMOLOGY

Let X be a smooth projective complex algebraic variety, and let $H_{\mathcal{M}}^{i,j}(X; R)$ denote its motivic cohomology with coefficients in a ring R . For every i and j , the topological realization functor $X \mapsto X(\mathbb{C})$ induces a natural homomorphism

$$(14) \quad H_{\mathcal{M}}^{i,j}(X; R) \rightarrow H^i(X; R)$$

to the singular cohomology of the space $X(\mathbb{C})$ of complex points of X .

For a prime p and integer $n \geq 0$, let $BP\langle n \rangle_{\mathcal{M}}^{i,j}(X)$ be the motivic Brown-Peterson cohomology for p and n constructed in [5, §6.4] and [10, §3]. Again, for every i and j , the topological realization functor from the motivic to the classical stable homotopy category induces a natural homomorphism

$$(15) \quad BP\langle n \rangle_{\mathcal{M}}^{i,j}(X) \rightarrow BP\langle n \rangle^i(X)$$

to the Brown-Peterson cohomology for p and n of the space $X(\mathbb{C})$.

Recall that in degrees $(2i, i)$, the group $H_{\mathcal{M}}^{2i,i}(X; \mathbb{Z})$ is naturally isomorphic to the Chow group $CH^i(X)$ of codimension i cycles on X . Therefore, we will denote the map (15) in degrees $(2i, i)$ by

$$\text{cl}_n: BP\langle n \rangle_{\mathcal{M}}^{2i,i}(X) \rightarrow BP\langle n \rangle^{2i}(X)$$

and extend this notation to the map (14) for mod p -cohomology

$$\text{cl}_{-1}: H_{\mathcal{M}}^{2i,i}(X; \mathbb{F}_p) \rightarrow H^{2i}(X; \mathbb{F}_p).$$

For every n and i , these maps fit into a commutative diagram

$$\begin{array}{ccc} BP\langle n \rangle_{\mathcal{M}}^{2i,i}(X) & \xrightarrow{\text{cl}_n} & BP\langle n \rangle^{2i}(X) \\ \rho_{-1, \mathcal{M}}^n \downarrow & & \downarrow \rho_{-1}^n \\ H_{\mathcal{M}}^{2i,i}(X; \mathbb{F}_p) & \xrightarrow{\text{cl}_{-1}} & H^{2i}(X; \mathbb{F}_p). \end{array}$$

Lemma 4.1. *Let $n \geq 0$ and X be a smooth projective complex variety. Let b_n be an element in $BP\langle n \rangle^*(X)$ which is not contained in the image of*

$$\rho_n^{n+1}: BP\langle n+1 \rangle^*(X) \rightarrow BP\langle n \rangle^*(X)$$

and has nontrivial image under ρ_{-1}^n in $H^(X; \mathbb{F}_p)$. Then b_n is not contained in the image of cl_n .*

Proof. Let $y_n \in H^*(X; \mathbb{F}_p)$ be the image of b_n under ρ_{-1}^n . If b_n was in the image of cl_n , then y_n was in the image of cl_{-1} as well. But, by the work of Totaro [19, Theorem 3.1] (see also Levine-Morel [9, Theorem 1.2.19]), the mod p -cycle map cl_{-1} factors through the natural map

$$BP^*(X) \otimes_{BP^*} \mathbb{Z}/p \rightarrow H^*(X; \mathbb{F}_p)$$

induced by $\rho_{-1}: BP^*(X) \rightarrow H^*(X; \mathbb{F}_p)$. Hence y_n would have to be contained in the image of ρ_{-1} . But this is impossible by our assumption that b_n is not contained in the image of ρ_n^{n+1} and hence not in the image of

$$\rho_n: BP^*(X) \rightarrow BP\langle n \rangle^*(X).$$

□

We can now extend the argument of Atiyah-Hirzebruch to all finite levels in the Brown-Peterson tower and prove our main result.

Theorem 4.2. *For every prime p and every integer $n \geq 0$, there is a smooth projective complex algebraic variety X and an element b_n in $BP\langle n \rangle^{2w(n)+2}(X)$ which is not contained in the image of the natural map*

$$\text{cl}_n: BP\langle n \rangle_{\mathcal{M}}^{2w(n)+2, w(n)+1}(X) \rightarrow BP\langle n \rangle^{2w(n)+2}(X).$$

Proof. Let $G_{n+3} = (\mathbb{Z}/p)^{n+3}$. Let k denote $2w(n+1) + 1$. By [15, §20] and [2, Proposition 6.6], there is a smooth projective variety X of complex dimension k together with a continuous map $X \rightarrow BG_{n+3} \times K(\mathbb{Z}, 2)$ which is k -connected. Let

$$\varphi: X \rightarrow BG_{n+3} \times K(\mathbb{Z}, 2) \rightarrow BG_{n+3}$$

denote the composition with the projection onto BG_{n+3} . For all $i \leq k$, any nonzero element $x \in H^i(G_{n+3}; \mathbb{F}_p)$ is pulled back to a nonzero element $\varphi^*(x) \in H^i(X; \mathbb{F}_p)$.

Let x_1, \dots, x_{n+3} denote the distinct exterior algebra generators of $H^*(G_{n+3}; \mathbb{F}_p)$ for p odd, or the distinct polynomial generators for $p = 2$. By Lemmas 3.2 and 3.6, we know that the element $Q_n \cdots Q_0(x_1 \cdots x_{n+3})$ is nontrivial in $H^{2w(n)+2}(G_{n+3}; \mathbb{F}_p)$. Thus, since $2w(n) + 2 \leq k$, the element

$$y_n := Q_n \cdots Q_0(\varphi^*(x_1 \cdots x_{n+3})) = \varphi^* Q_n \cdots Q_0(x_1 \cdots x_{n+3}) \in H^{2w(n)+2}(X; \mathbb{F}_p)$$

is nontrivial as well. Now we define

$$b_n := q_n \cdots q_0(\varphi^*(x_1 \cdots x_{n+3})) = \varphi^* q_n \cdots q_0(x_1 \cdots x_{n+3}) \in BP\langle n \rangle^{2w(n)+2}(X).$$

Since its image under the canonical map

$$BP\langle n \rangle^{2w(n)+2}(X) \rightarrow H^{2w(n)+2}(X; \mathbb{F}_p)$$

is $\pm y_n$, b_n is nontrivial as well. Hence, by Lemma 2.1, b_n is not contained in the image of $\rho_n^{n+1}: BP\langle n+1 \rangle^*(X) \rightarrow BP\langle n \rangle^*(X)$. By Lemma 4.1 this implies that b_n is not contained in the image of cl_n . \square

Remark 4.3. The minimal possible (complex) dimension of the variety in Theorem 4.2 is $2w(n+1) + 1 = 2(p^{n+1} + \cdots + 1) + 1$. This is because we need the map $X \rightarrow BG_{n+3} \times K(\mathbb{Z}, 2)$ to be $2w(n+1) + 1$ -connected for the argument to work. Hence, for $p = 2$, we see that the least possible dimension of X is $2^{n+3} - 1$. The fact that the dimension is rather big is consistent with a result of Soulé-Voisin [16, Theorem 1] that the order of an element detected by the Atiyah-Hirzebruch-Totaro obstruction must be small relative to the dimension of the variety.

Remark 4.4. The case $n = 0$ of Theorem 4.2 is the original example of Atiyah and Hirzebruch [2, p. 42, Proof of (6.7)]. Let $G_3 = \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$ with cohomology ring $H^*(G_3; \mathbb{F}_p) = \Lambda(x_1, x_2, x_3) \otimes \mathbb{F}_p[y_1, y_2, y_3]$. Atiyah and Hirzebruch consider the element $y := \beta(x_1 x_2 x_3) \in H^4(G_3; \mathbb{F}_p)$, where β denotes the Bockstein homomorphism, and show $Q_1(y) \neq 0$ in $H^{4+2p-1}(G_3; \mathbb{F}_p)$. Since y is in the kernel of β , it corresponds to a unique p -torsion element in $H^*(G_3; \mathbb{Z})$. The Godeaux-Serre construction then provides a smooth projective variety X whose cohomology contains $H^*(G_3; \mathbb{Z})$ as a direct factor up to degree $\dim_{\mathbb{C}} X$. Then they show that a class in the image of cl must be a permanent cycle in the Atiyah-Hirzebruch spectral sequence $H^*(X; \mathbb{Z}) \Rightarrow KU^*(X)$ converging to the complex K -theory of X . The operation Q_1 corresponds to a differential in this spectral sequence. Hence if Q_1 acts non-trivially on y , then the lift of y to $H^*(X; \mathbb{Z})$ cannot be a permanent cycle.

Remark 4.5. The elements constructed in Theorem 4.2 map to torsion classes in the $\mathbb{Z}_{(p)}$ -cohomology of $X(\mathbb{C})$. In particular, their images in $H^*(X; \mathbb{Z}_{(p)})$ are $\mathbb{Z}_{(p)}$ -Hodge classes.

Remark 4.6. Let $\Omega^*(-)$ denote the algebraic cobordism theory of Levine and Morel [9]. It is constructed as the universal oriented cohomology theory for smooth algebraic varieties over fields of characteristic 0. For smooth varieties over \mathbb{C} , it comes equipped with a natural commutative diagram

$$(16) \quad \begin{array}{ccc} \Omega^*(X) & \xrightarrow{\text{cl}_\Omega} & MU^{2*}(X) \\ \theta \downarrow & & \downarrow \vartheta \\ CH^*(X) & \xrightarrow{\text{cl}} & H^{2*}(X; \mathbb{Z}). \end{array}$$

One might wonder what the image of cl_Ω might be. A first restriction is given by diagram (16) itself. For the image of cl_Ω has to be contained in the subgroup $\text{Hdg}_{MU}^{2*}(X)$ of elements in $MU^{2*}(X)$ which are mapped to Hodge classes under ϑ . But, as Claire Voisin kindly pointed out to us, one can say more. As in Kollár's example, let X be a very general smooth hypersurface in \mathbb{P}^4 of degree divisible by s^3 for an integer s coprime to 6. Then the map ϑ is surjective and all integral cohomology classes in $H^4(X; \mathbb{Z})$ are Hodge classes. Thus, since there is a class $\alpha \in H^4(X; \mathbb{Z})$ which is not contained in the image of cl , there is an element in $MU^4(X)$ mapping to α which is not contained in the image of cl_Ω .

Remark 4.7. In [4], Colliot-Thélène and Szamuely use the examples of [2] to show that the ℓ -adic integral version of the Tate conjecture for varieties over finite fields of characteristic $\neq \ell$ fails (in [13], one can find a different explanation of the obstructions). More recently, it was shown by Pirutka-Yagita [11] for the primes $\ell = 2, 3$ or 5 , and by Kameko [7] for all primes ℓ , that the integral version of the Tate conjecture even fails for non-torsion classes. Those examples are based on the examples by Atiyah-Hirzebruch and approximations of projective varieties by classifying spaces of affine algebraic groups. In [1], Antieau shows that there is another type of non-torsion examples which come from classifying spaces of quotients of special linear groups. The argument uses representation theory, while the Q_i vanish on those examples. However, there are differentials of higher degree in the Atiyah-Hirzebruch spectral sequence that do detect the examples.

Remark 4.8. By [15, §20], Godeaux-Serre varieties can be constructed over any infinite field. In particular, we could use the techniques of [13] and consider a smooth projective variety X over an algebraically closed field \mathbb{F}_ℓ of characteristic $\ell \neq p$. There is a p -completed étale version of Brown-Peterson cohomology for p and n , denoted by $\hat{BP}\langle n \rangle_{\text{ét}}^*(X)$. Moreover, the results of [5] allow to construct a motivic version $BP\langle n \rangle_{\mathcal{M}}^{2*,*}(X)$ in characteristic $\ell \neq p$. The stable p -completed étale realization functor then induces a natural map

$$BP\langle n \rangle_{\mathcal{M}}^{2*,*}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \hat{BP}\langle n \rangle_{\text{ét}}^{2*}(X).$$

Then one can use Milnor operations in étale cohomology and repeat our argument for Godeaux-Serre varieties defined over \mathbb{F}_ℓ . This yields non-algebraic classes in the étale Brown-Peterson tower in characteristic $\ell \neq p$.

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